Calculus. Limits and derivatives

Secants and cosecants (definitions)

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x}$$

Comparative growth of sequences at $n \to +\infty$

$$\boxed{\ln n \ll n^c \ll a^n \ll n! \ll n^n}$$ (conditions: $c > 0$, $a > 1$)

Continuity at a point (definition)

$$f(x) \in C(x_0) \Leftrightarrow \lim_{x \to x_0^-} f(x) = f(x_0) = \lim_{x \to x_0^+} f(x)$$

The first special limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Corollaries

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\arcsin x}{x} = \lim_{x \to 0} \frac{\arctan x}{x} = 1 \qquad \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

The second special limit

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

Exponential function (definition)

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

Double inequality used in the sandwich (squeeze) theorem with the second special limit

$$\boxed{\frac{x}{1+x} \le \ln(1+x) \le x} \qquad \frac{1}{1+x} \le \ln\left(1 + \frac{1}{x}\right) \le \frac{1}{x}$$

Corollaries

$$\lim_{x \to 0} (1 + ax)^{\frac{1}{x}} = e^a \qquad \lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 0 \qquad \lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$$

Equivalent functions at $x \to 0$ (definition)

$$f(x) \sim g(x) \Leftrightarrow \begin{cases} f(x) = g(x) \cdot \alpha(x) \\ \lim_{x \to 0} \alpha(x) = 1 \end{cases}$$

Table of equivalent functions at $x \to 0$

$$\sin x \sim x \qquad \cos x \sim 1 - \frac{x^2}{2}$$

$$\tan x \sim x \qquad e^x \sim 1 + x$$

$$\arcsin x \sim x \qquad a^x \sim 1 + x \ln a$$

$$\arctan x \sim x \qquad \boxed{(1+x)^k \sim 1 + kx}$$

$$\ln(1+x) \sim x$$

Slant asymptote of the graph $y = f(x)$ is a line $y = kx + b$ with the following coefficients

$$k = \lim_{x \to \infty} \frac{f(x)}{x} \qquad b = \lim_{x \to \infty} (f(x) - kx)$$

The Intermediate Value Theorem. A function continuous on a closed interval *attains all the intermediate values* between its values at the endpoints.
The Root Theorem. If a function continuous on a closed interval attains the values of opposite signs at the endpoints, then it must have *at least one root* inside this interval.
The Extreme Value Theorem. A function continuous on a closed interval *attains its minimal and its maximal values* on this interval.
The Boundedness Theorem. A function continuous on a closed interval is *bounded*.

Derivative (definition)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Differential (definition)

$$df(x) = f'(x)dx$$

Linear approximation

$$f(x+dx) = f(x) + df(x) = f(x) + f'(x)dx \quad \Rightarrow \quad \boxed{\boldsymbol{f(x+\Delta x) \approx f(x) + f'(x)\Delta x}}$$

Criterion of differentiability

$f(x)$ is differentiable at a point $x = x_0$ if it is continuous at $x = x_0$ and side derivatives at $x = x_0$ are equal, i.e. $f_-{}'(x_0) = f_+{}'(x_0)$.

The chain rule

$$\left(f\left(g(x)\right)\right)' = f'(g(x)) \cdot g'(x)$$

Notation $(h(x))|_{x=s(x)}$ means that variable x in the function $h(x)$ should be substituted by the function $s(x)$. So, the chain rule could be rewritten as following: $\left(f\left(g(x)\right)\right)' = (f'(x))|_{x=g(x)} \cdot g'(x)$.

Partial derivative (notation)

$$\partial_x F(x,y) = \frac{\partial F(x,y)}{\partial x}$$

Derivative of an implicit function y given by an implicit equation $F(x,y) = 0$

$$\boxed{\boldsymbol{y'(x) = -\frac{\partial_x F(x,y)}{\partial_y F(x,y)}}}$$

Example from microeconomics. If $u(X,Y)$ is a utility function (that means $u(X,Y) = c$ is an equation of an indifference curve for any constant c), $MU_X = \partial_X u(X,Y)$ and $MU_Y = \partial_Y u(X,Y)$ are functions of marginal utility of each good, then marginal rate of substitution of good X for good Y is $MRS_{XY} = -\frac{dY}{dX} = \frac{\partial_X u(X,Y)}{\partial_Y u(X,Y)} = \frac{MU_X}{MU_Y}$.

Inverse function $f^{-1}(x)$ (definition)

$$f^{-1}\left(f(x)\right) = x$$

Derivative of the inverse function

$$\boxed{\boldsymbol{\left(f^{-1}(x)\right)'\Big|_{x=x_0} = \frac{1}{f'\left(f^{-1}(x_0)\right)}}}$$

Logarithmic derivative

$$(\ln f(x))' = \frac{f'(x)}{f(x)}$$

Equation of a tangent line to the graph $y = f(x)$ at the point x_0

$$t(x) = f'(x_0) \cdot (x - x_0) + f(x_0)$$

Equation of a normal line to the graph $y = f(x)$ at the point x_0

$$n(x) = -\frac{1}{f'(x_0)} \cdot (x - x_0) + f(x_0)$$

Average rate of change of the function $f(x)$ on $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}$$

Clarification. $f(x)$ should be continuous on $[a, b]$ and differentiable on (a, b).

The Mean Value Theorem (Lagrange Theorem)

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b).
Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The Rolle's Theorem (a special case of MVT)

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$.
Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

The L'Hospital's Rule

If **1)** $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \vee \pm\infty$, **2)** $f(x)$, $g(x)$ are differentiable around point $x = a$, **3)** $g'(x) \neq 0$ around point $x = a$, then

$$\frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{\lim_{x \to a} f'(x)}{\lim_{x \to a} g'(x)}$$